Performance Analysis of the M/M/2/2 System with Heterogeneous Servers

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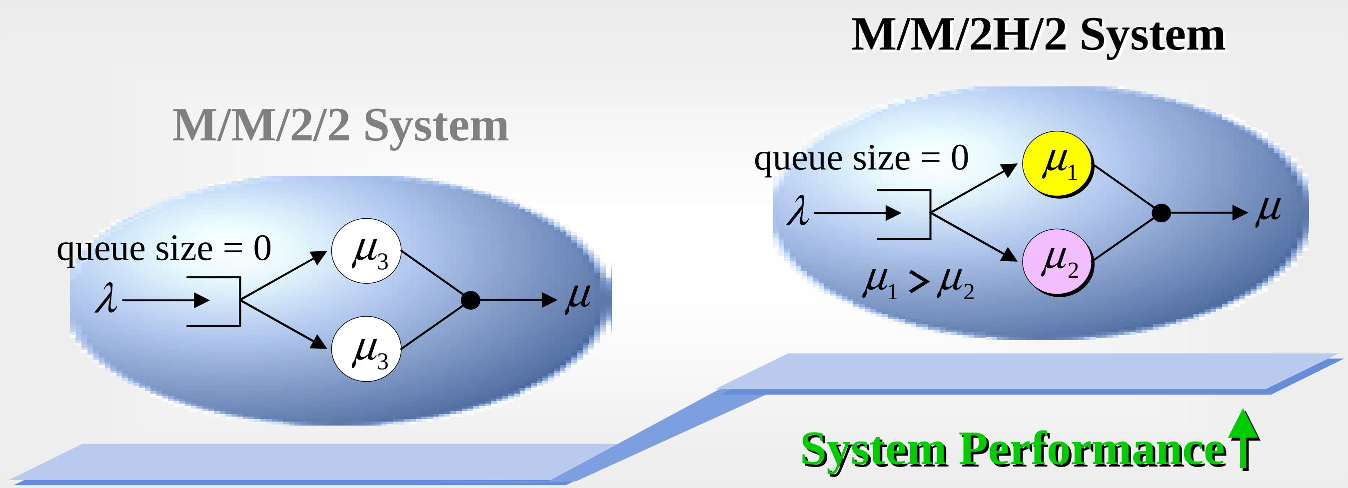
Research Objective

- To improve the System Performance of the M/M/2/2 System by Replacing its Homogeneous Servers with Heterogeneous Servers
- To identify the effect of the service ratio (β) on the system performance

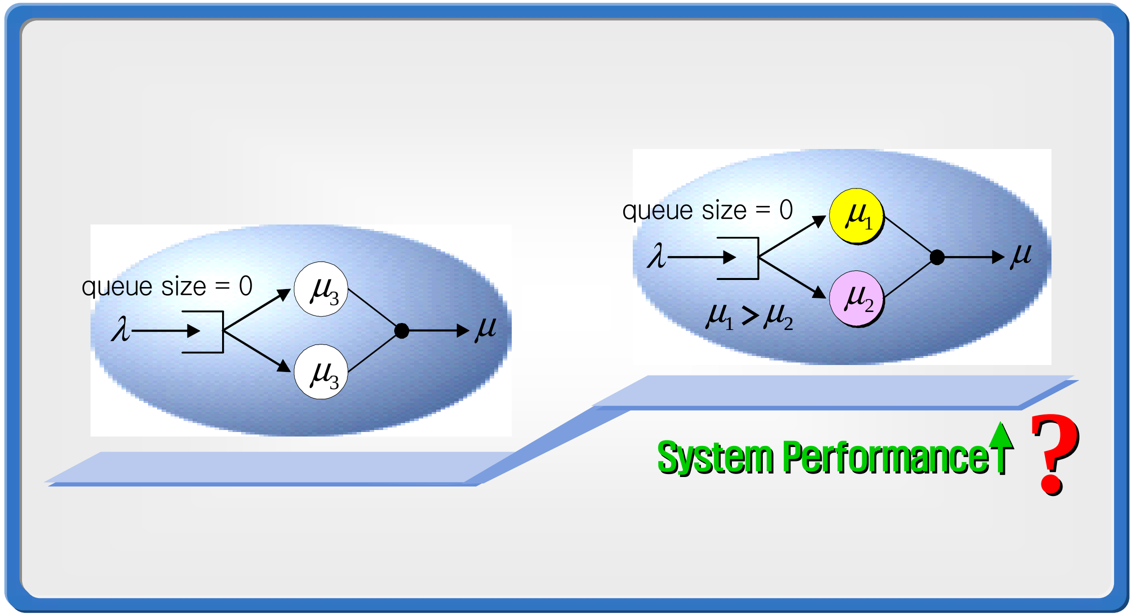


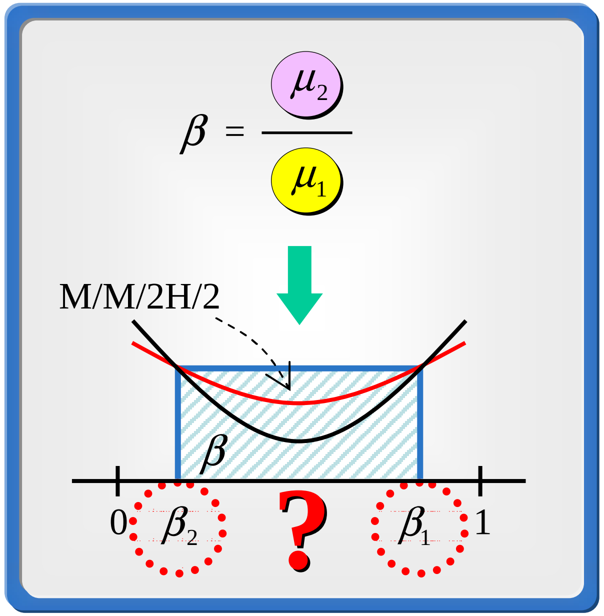
INTRODUCTION

Detailed Objectives

- To show the Superiority of the M/M/2H/2 system over M/M/2/2

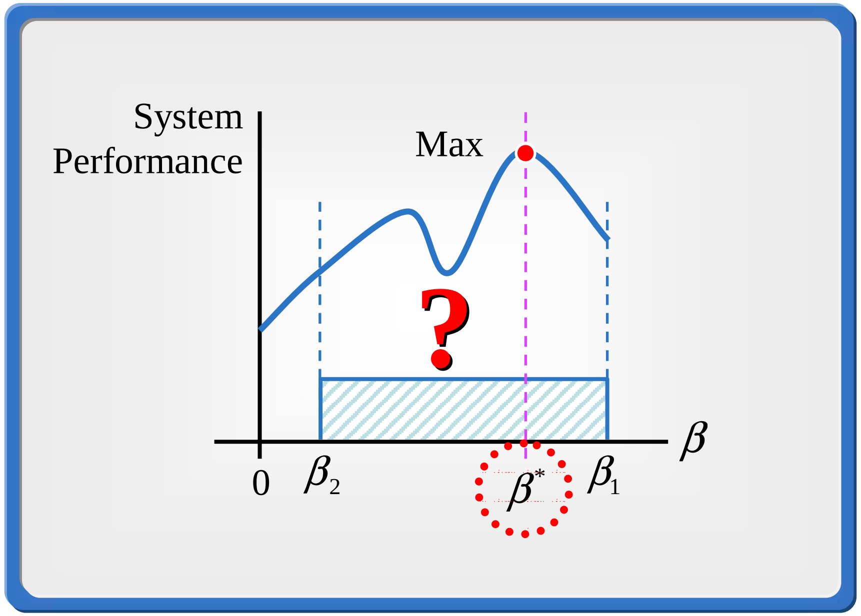
  ➔ Existence of the Service Ratio between two heterogeneous servers that makes the M/M/2H/2 system outperform over M/M/2/2

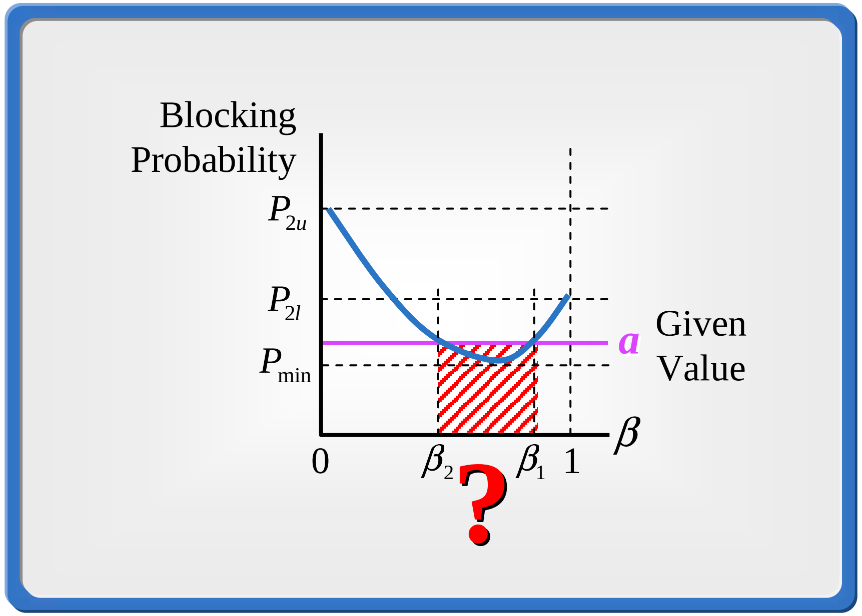




Detailed Objectives

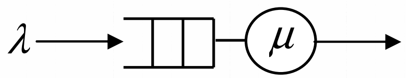
- To show the Existence of the Optimal Service Ratio
- To show the Controlability of the System Performance





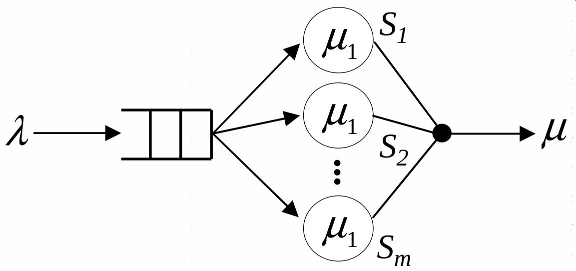
BACKGROUND OF THIS STUDY

M/M/1 System



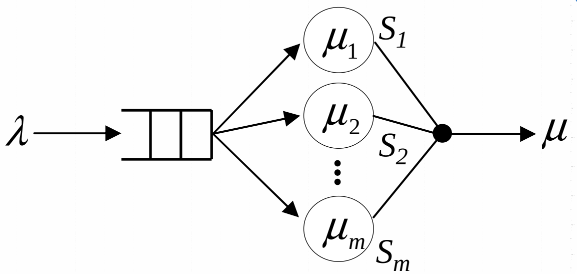
- System performance is determined by the fixed service rate
- No sequencing problem due to FCFS service rule

M/M/m System



- Enhanced performance by multiple homogeneous servers
- Service rule: No preference among *m* servers
  → The packet at the head of queue is assigned to a server as soon as a server becomes idle
- Re-sequencing (reordering) problem

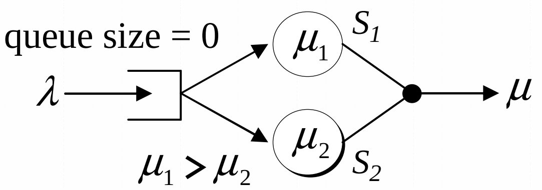
M/M/mH System



- Enhanced performance by multiple heterogeneous servers
- Service rule: Preference is given to the fastest server
- Re-sequencing (reordeing) problem
- A Two-Server System (M/M/2H) is considered ***PRAGMATIC*** !!
  ➡ M/M/2H/2 : Queue-less M/M/2H system

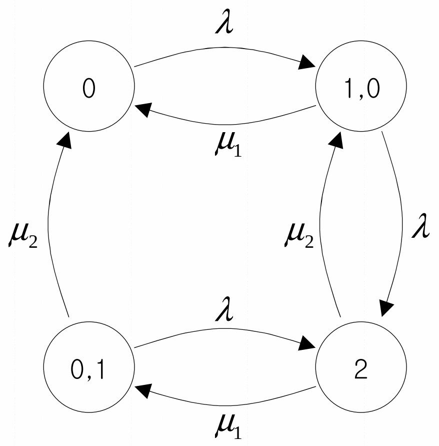
ANALYSIS OF THE M/M/2H/2 SYSTEM

M/M/2H/2 System



- No queue space
- Service Rule: Preference is given to fast server S1
  1. **If both servers are idle, a new arrival is sent to server S1.**
  2. If one of the two servers is idle, a new arrival is sent to the idle server.
  3. If both servers are busy, a new arrival is blocked and lost.

State Transition Diagram of the M/M/2H/2 System



- N : System Size → When N =1, (N_1, N_2) = (1, 0) or (0, 1)
- State Space ={$k \in$ 0, (1,0), (0,1), 2}
- P_k: Probability of state k

$$\left.\begin{aligned} \lambda P_0 &= \mu_1 P_{10} + \mu_2 P_{01} \\ (\lambda+\mu_1)P_{10} &= \lambda P_0 + \mu_2 P_2 \\ (\lambda+\mu_2)P_{01} &= \mu_1 P_2 \\ (\mu_1+\mu_2)P_2 &= \lambda(P_{01}+P_{10}) \end{aligned}\right\} \xrightarrow[\mu=\mu_1+\mu_2]{P_2} \left\{\begin{aligned} P_0 &= \frac{\mu_1\mu_2 P_2}{\lambda(\lambda+\mu_2)}\left(\frac{\mu}{\lambda}+2\right) \\ P_{01} &= \frac{\mu_1 P_2}{\lambda+\mu_2} \\ P_{10} &= \frac{\mu_2(\lambda+\mu)P_2}{\lambda(\lambda+\mu_2)} \end{aligned}\right.$$

ANALYSIS OF THE M/M/2H/2 SYSTEM

Erlang B Probability of the M/M/2H/2 System

- Using P_0, P_{10}, P_{01}, and the identity equation $P_0+P_{10}+P_{01}+P_2=1$,
- Utilization factor : $\rho = \lambda/(\mu_1+\mu_2)$
- Service ratio : $\beta = \mu_2/\mu_1 \quad 0<\beta<1$

$$P_2 = \frac{\rho^2\left[(1+\rho)\beta^2+(1+2\rho)\beta+\rho\right]}{\rho(1+\rho)^2\beta^2+(1+2\rho)(1+\rho+\rho^2)\beta+\rho^2(1+\rho)}$$

n-th Moment of the System Size *N*

- *n*-th Moment of the System Size N : $E(N^n) = P_{10}+P_{01}+2^n P_2 = \left(2^n+\frac{1}{\rho}\right)P_2$
- Mean system size $E(N)$: $E(N) = \frac{\rho(1+2\rho)\left[(1+\rho)\beta^2+(1+2\rho)\beta+\rho\right]}{\rho(1+\rho)^2\beta^2+(1+2\rho)(1+\rho+\rho^2)\beta+\rho^2(1+\rho)}$

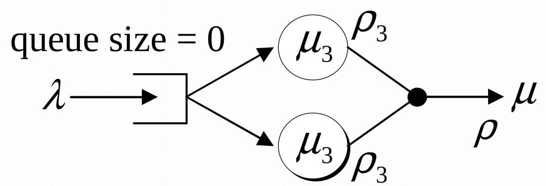
SUPERIORITY OF THE M/M/2H/2 SYSTEM

Erlang B Probability of the M/M/2/2 System

- Erlang B Prob. of the M/M/2/2 system : $P_b = \dfrac{\rho_3^2}{2+2\rho_3+\rho_3^2}$

  where ρ_3 is the utilization factor of each server used in the M/M/2/2 system.

- Since $\mu = 2\mu_3$,
  we have $\rho_3 = \lambda/\mu_3 = 2\lambda/\mu = 2\rho.$ ➡ $P_b = \dfrac{2\rho^2}{1+2\rho+2\rho^2}$



Superiority in terms of the Erlang B Probability

- In order to show the superiority of the M/M/2H/2 system over the M/M/2/2 system, let $P_2 < P_b$. Then

$$\frac{P_2}{P_b} = \frac{(1+2\rho+2\rho^2)\left[(1+\rho)\beta^2+(1+2\rho)\beta+\rho\right]}{2\left[\rho(1+\rho)^2\beta^2+(1+2\rho)(1+\rho+\rho^2)\beta+\rho^2(1+\rho)\right]} < 1$$

$$\Rightarrow \quad (1+\rho)\beta^2-(1+2\rho)\beta+\rho=(1+\rho)(\beta-\beta_1)(\beta-\beta_2)<0$$

$$\therefore \beta_1 = 1 \text{ and } \beta_2 = \rho/(1+\rho) \quad \Rightarrow \quad \beta_2 < \beta < \beta_1$$

➡ **There exists an interval of such β that $P_2 < P_b$**

(Erlang B probability of the M/M/2H/2 system is smaller than that of the M/M/2/2 system)

SUPERIORITY OF THE M/M/2H/2 SYSTEM

Superiority in terms of the Mean System Size

- Mean System Size of the M/M/2/2 system : $E_b(N^n) = \left(2^n + \frac{1}{\rho}\right)P_b$
- Mean System Size of the M/M/2H/2 system : $E(N^n) = P_{10} + P_{01} + 2^n P_2 = \left(2^n + \frac{1}{\rho}\right)P_2$
- Let $E(N^n) < E_b(N^n)$, then we have straightforwardly $P_2 < P_b$.

➡ **There exists an interval of such β that $E(N^n) < E_b(N^n)$**

(Mean System Size of the M/M/2H/2 system is smaller than that of the M/M/2/2 system)

Superiority in terms of the Mean Sojourn Time

- Mean Sojourn Time of the M/M/2/2 System : $E_b(T) = E_b(N)/\lambda$
- Mean Sojourn Time of the M/M/2H/2 System : $E(T) = E(N)/\lambda$
- Let $E(N) < E_b(N)$, then we have straightforwardly $P_2 < P_b$.

➡ **There exists an interval of such β that $E(N) < E_b(N)$**

(Mean Sojourn Time of the M/M/2H/2 system is smaller than that of the M/M/2/2 system)

Existence of the Optimal β

- Optimal β means the value of β which minimizes the Erlang B probability, P_2, of the M/M/2H/2 system.
- In order to find such β, we differentiate P_2 in terms of β. Then, we have

$$\frac{dP_2}{d\beta} = \frac{\rho^2(1+2\rho)\left[(1+\rho)\beta^2 - \rho\right]}{\left[\rho(1+\rho)^2\beta^2 + (1+2\rho)(1+\rho+\rho^2)\beta + \rho^2(1+\rho)\right]^2}$$

- Let $\frac{dP_2}{d\beta} = 0$ then we have $\beta = \beta^* = \sqrt{\frac{\rho}{1+\rho}}$

- Tis stationary point is the minimizer of P_2 because P_2 is convex for $\beta \in [0,1]$

$$\frac{d^2P_2}{d\beta^2} = \frac{2(1+2\rho)\rho^3}{D^3}\left[1-\beta^3+3(1+\beta-\beta^3)+3(1+2\beta-\beta^3)\rho^2+(2+3\beta-\beta^3)\rho^3\right] > 0$$

where $D = \rho(1+\rho)^2\beta^2 + (1+2\rho)(1+\rho+\rho^2)\beta + \rho^2(1+\rho)$.

- Therefore, the minimum of P_2 is obtained by substituting β^* into P_2.

$$P_2^* = \frac{\rho^2\left[1+2\rho+2\sqrt{\rho(1+\rho)}\right]}{(1+2\rho)(1+\rho+\rho^2)+2\rho(1+\rho)\sqrt{\rho(1+\rho)}}$$

➡ **There exists an optimal β that minimizes the Erlang B Prob. of the M/M/2H/2 system**

Control of the Erlang B Probability with β

- How to determine an interval of β such that the Erlang B Prob. is less then a predefined value *a*?

$$\frac{\rho^2\left[(1+\rho)\beta^2+(1+2\rho)\beta+\rho\right]}{\rho(1+\rho)^2\beta^2+(1+2\rho)(1+\rho+\rho^2)\beta+\rho^2(1+\rho)}<a$$

- *a* should be feasible: $a \geq P_2^*$

$$\Rightarrow\ \left[\rho-a(1+\rho)\right]\rho^2+(1+2\rho)\left[\rho^2-a(1+\rho+\rho^2)\right]\beta+\rho(1+\rho)\left[\rho-a(1+\rho)\right]\beta^2<0$$

- The solution of above equation is

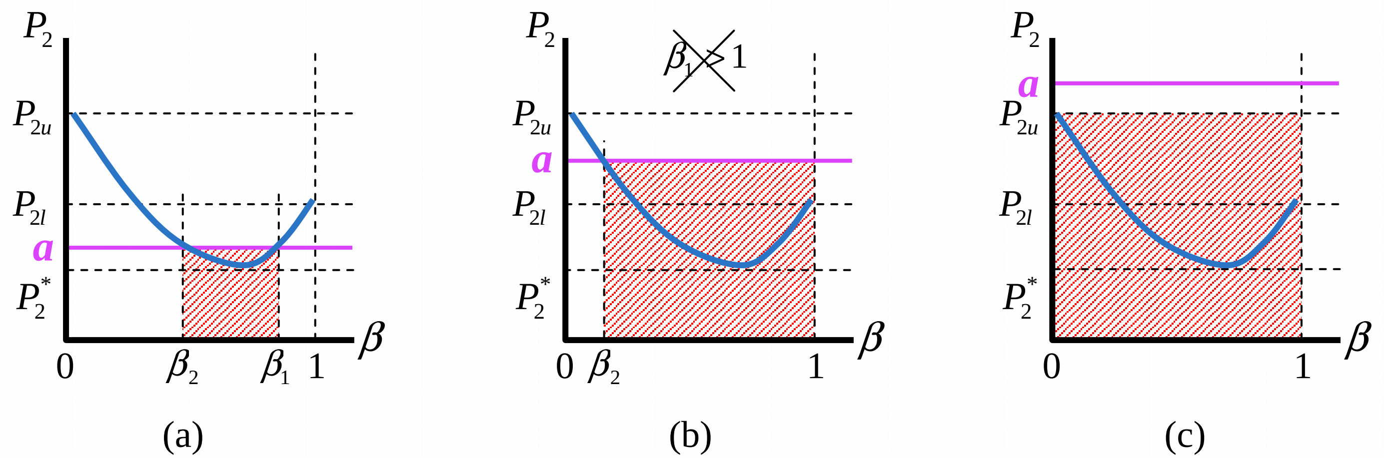
$$\rho(1+\rho)\left[\rho-a(1+\rho)\right](\beta-\beta_1)(\beta-\beta_2)<0$$

where $\beta_1=\dfrac{(1+2\rho)\left[a(1+\rho+\rho^2)-\rho^2\right]+\sqrt{D_\beta}}{2\rho(1+\rho)\left[\rho-a(1+\rho)\right]}$ and $\beta_2=\dfrac{(1+2\rho)\left[a(1+\rho+\rho^2)-\rho^2\right]-\sqrt{D_\beta}}{2\rho(1+\rho)\left[\rho-a(1+\rho)\right]}$

and $D_\beta=(1+2\rho)^2\left[\rho^2-a(1+\rho+\rho^2)\right]^2-4\rho^3(1+\rho)\left[\rho-a(1+\rho)\right]^2 \quad (D_\beta \geq 0)$

Control of the Erlang B Probability with β

- Let $\lim_{\beta \to 0} P_2 = \frac{\rho}{1+\rho} = P_{2u}$ and $\lim_{\beta \to 1} P_2 = \frac{2\rho^2}{1+2\rho+2\rho^2} = P_{2l} < P_{2u}$
- Consequently,
  - (a) If $P_2^* < a \le P_{2l}$ then $\beta_2 < \beta < \beta_1$
  - (b) If $P_{2l} < a \le P_{2u}$ then $\beta_2 < \beta < 1$
  - (c) If $P_{2u} < a$ then $0 < \beta < 1$



SUMMARY AND REMARKS

Summary

- The M/M/2H/2 system was Defined and Analyzed
- Superiority of the M/M/2H/2 system over the M/M/2/2 system
- Existence of the Optimal β
- Control of the Erlang B Probability with β

SUMMARY

Remarks

- The system could cause the re-sequencing/reordering problem
- Applications: Telephony system only?

REMARKS

Thank you!

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